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TOPOLOGY AND THE DUALS OF CERTAIN LOCALLY COMPACT GROUPS

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Abstract. We consider some topological questions concerning the dual space of a (separable) extension G of a type I, regularly embedded subgroup N. The dual $\hat{G}$ is known to have a fibre-like structure. The fibres are in bijective correspondence with certain subsets of dual spaces of associated stability subgroups. These subsets in turn are in bijective correspondence with certain projective dual spaces. Under varying hypotheses, we give sufficient conditions for these bijections to be homeomorphisms, we determine the support of the induced representation U^L (for $L \in \hat{N}$) and we give necessary and sufficient conditions for a union of fibres in $\hat{G}$ to be closed.

In a much more general context we study the Hausdorff and CCR separation properties of the dual of an extension. We then completely describe the dual space topology of the above extension G in an interesting case.

The preceding results are then applied to the case where N is abelian and G/N is compact.

1. **Introduction.** In this article, we consider some topological questions concerning the dual space $\hat{G}$ of a (separable) locally compact extension G of a type I, regularly embedded subgroup N.

The dual $\hat{G}$ is known to have a fibre-like structure over a space of orbits θ. With the notation as in §3, we first show that a fibre $\hat{G}_\theta$ will be homeomorphic to the corresponding subset $\mathcal{M}_L$ (for $L \in \theta$) of the dual $\hat{H}_L$ of the associated stability subgroup H_L when this subgroup is open. Secondly, $\mathcal{M}_L$ is in turn homeomorphic to its associated projective dual space $(H_L/N, \sigma_L)^\wedge$ when L is one-dimensional. We then determine the support in $\hat{G}$ of the induced representation U^L when H_L/N is an R-group(1). Finally, if N is CCR and G/N is an R-group, then a union of fibres in $\hat{G}$ is closed precisely when the underlying union of orbits is closed.

In §4 we give sufficient conditions for a group extension to not have a Hausdorff dual and for an induced representation to be CCR. Specifically, in the latter case we show that if T is a CCR representation of the closed subgroup H with compact quotient space, then U^T is CCR. This implies that a compact extension of a CCR group is CCR.

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(1) This is the same as an amenable group.

We completely determine (§5) the dual space topology of the group extension G described at the beginning when the nonidentity stability subgroups are the same group H and G/H is an R-group (i.e. the "essentially free" case).

§6 is devoted to applying the preceding results to the collection of (separable) groups which are compact extensions of abelian groups.

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2. **Preliminaries.** All groups considered will be separable and locally compact and all representations will be strongly continuous, unitary and of countable dimension. We will assume the reader is familiar with the notion and properties of weak containment as developed by J. M. G. Fell [7]–[12].

If σ is a multiplier for the group G, then Rep (G, σ) will denote the set of (unitary) equivalence classes of (separable) σ-representations of G. We will not distinguish between a representation and its equivalence class. When σ is trivial we will delete it. We consider Rep (G) to be equipped with the inner hull-kernel topology [9, §2] and the dual $\hat{G}$ to be equipped with its usual hull-kernel topology [7, p. 372]. The notion of weak containment is also meaningful for projective representations [2, §3]. Thus, we may consider the σ-dual $(G, \sigma)^\wedge$ (respectively Rep (G, σ)) to be equipped with its corresponding hull-kernel (respectively inner hull-kernel) topology.

The following lemma is basic to what follows.

LEMMA 2.1. *Let G and H be groups with multipliers σ and τ respectively. Suppose $\mathcal{T}$ is a subset of $(G, \sigma)^\wedge$ with induced topology. If $\varphi: \mathcal{T} \to (H, \tau)^\wedge$, then the following are equivalent:*

(i) *φ is continuous.*

(ii) *If $T \in \mathcal{T}$, $\mathcal{S} \subseteq \mathcal{T}$ and T is weakly contained in $\mathcal{S}$, then $\varphi(T)$ is weakly contained in $\varphi(\mathcal{S})$.*

Proof. This is a direct consequence of the definition of hull-kernel topology.

3. **Some topological properties.** Let G be a group, N a closed normal subgroup which is type I and regularly embedded [19, p. 302]. We then have a mapping $G \times \hat{N} \to \hat{N}$ given by $(y, L) \to L^y$, where $L^y_x = L_{yxy^{-1}}$, $x \in N$. For each $L \in \hat{N}$ we let $\theta(L)$ be the orbit $\{L^y : y \in G\}$. The orbits partition $\hat{N}$ and give rise to the orbit space $\hat{N}/G$. If $L \in \hat{N}$, then $H_L = \{y \in G : L^y = L\}$ is a closed subgroup of G containing N [19, Theorem 7.5] which we call the stability subgroup of L. Let $\pi_L: H_L \to H_L/N$ be the corresponding canonical homomorphism. For each $L \in \hat{N}$, let $\mathcal{M}_L$ denote the set of T in $\hat{H}_L$ such that the restriction $T|N$ is a multiple of L. If σ_L is a multiplier for H_L/N, then $\sigma_L \pi_L$ is a multiplier for H_L as is its inverse $(\sigma_L \pi_L)^{-1}$.

MACKEY'S THEOREM. *Let the notation be as above. There exists an onto mapping $p: \hat{G} \to \hat{N}/G$ such that if $\hat{G}_\theta = p^{-1}(\theta)$, $\theta \in \hat{N}/G$, then the mapping $T \to U^T$ is a bijection between $\mathcal{M}_L$ and $\hat{G}_\theta$, where L is any element of the orbit θ. The mapping p is* *given by* $p(U^T)=\theta$. *Furthermore, there exists a unique* (*up to similarity* [19, *p.* 269]) *multiplier* σ_L *for* H_L/N *such that* L *extends to a* $(\sigma_L\pi_L)^{-1}$—*representation* L' *of* H_L *and the mapping* $M \to L' \otimes M\pi_L$ *is a bijection between* $(H_L/N, \sigma_L)^\wedge$ *and* $\mathcal{M}_L$.

LEMMA 3.1. *The mapping* p *is continuous* (*where* $\hat{N}/G$ *is given the quotient topology it inherits from* $\hat{N}$).

Proof. This is Lemma 3 of [10].

Let $\theta \in \hat{N}/G$ and $L \in \theta$. We will refer to $\hat{G}_\theta$ as the fibre over θ. The sets $\hat{G}_\theta$ and $\mathcal{M}_L$ are equipped with induced topologies as subsets of $\hat{G}$ and $\hat{H}_L$ respectively. We first investigate the topological properties of the bijection $M \to L' \otimes M\pi_L$.

PROPOSITION 3.1. *Let* G *be a group,* σ *and* τ *multipliers for* G, $V \in \operatorname{Rep}(G, \sigma)$, $T \in \operatorname{Rep}(G, \tau)$ *and* $\mathcal{S} \subseteq \operatorname{Rep}(G, \tau)$. *If* T *is weakly contained in* $\mathcal{S}$, *then the* $\sigma\tau$-*representation* $V \otimes T$ *is weakly contained in the* $\sigma\tau$-*representations*

$$\{V \otimes S : S \in \mathcal{S}\} = V \otimes \mathcal{S}.$$

Proof. We can verify directly that each fundamental function associated with $V \otimes T$ [2, p. 278] which is obtained from a generator $v \otimes w$ (for $v \in \mathcal{H}(V)$ and $w \in \mathcal{H}(T)$) is a uniform-on-compacta limit of finite sums of fundamental functions associated with $V \otimes \mathcal{S}$. Now apply [11, Lemma 1].

This proposition is valid for nonseparable groups as well and combines the advantages of [9, p. 260] and [2, p. 288]. We have the following partial converse.

LEMMA 3.2. *If* V *is one-dimensional, then* $V \otimes T$ *is weakly contained in* $V \otimes \mathcal{S}$ *implies* T *is weakly contained in* $\mathcal{S}$.

Proof. Straightforward.

PROPOSITION 3.2. *Let the notation be as in Mackey's Theorem. Then the bijection* $M \to L' \otimes M\pi_L$ *of* $(H_L/N, \sigma_L)^\wedge$ *onto* $\mathcal{M}_L$ *is continuous. If* L *is one-dimensional, then this bijection is a homeomorphism.*

Proof. The first assertion is clear. The second is also clear once we observe that M is weakly contained in $\mathcal{S} \subseteq (H_L/N, \sigma_L)^\wedge$ if and only if $M\pi_L$ is weakly contained in $\{S\pi_L : S \in \mathcal{S}\}$.

Let us now consider the topological properties of the bijection $T \to U^T$ between $\hat{G}_\theta$ and $\mathcal{M}_L$, for $\theta \in \hat{N}/G$, $L \in \theta$. This mapping is continuous [9, Theorem 4.2] and a homeomorphism if H_L is normal in G ([10, Lemma 4] and [15, Theorem 1]). The same is true if H_L is open in G. We require some preliminary information before we can prove this.

Let G be a group, H an open subgroup and T a representation of H. For each $y \in G$, we form the representation T^y of the open subgroup $y^{-1}Hy \cap H$ by defining $T^y_x = T_{yxy^{-1}}$, $x \in y^{-1}Hy \cap H$. Induce T^y up to H and denote the resulting representation of H by $V^{T,y}$. Note in particular that $V^{T,e}=T$. Denote $\{V^{T,y} : y \in G\}$ by $V^{T,G}$.

LEMMA 3.3. *$V^{T,y}$ depends only on the double coset HyH to which y belongs. If we induce T to G to get U^T, then $U^T|H$ is a direct sum of the representations $V^{T,G}$.*

Proof. This is a special case of [18, Theorem 7.1].

LEMMA 3.4. *$U^T|H$ is weakly equivalent to $V^{T,G}$.*

Proof. This is a special case of [9, Theorem 3.2].

THEOREM 3.1. *Let the notation be as in Mackey's Theorem. If H_L is an open subgroup of G, then $\mathcal{M}_L$ is homeomorphic to $\hat{G}_\theta$ under the mapping $T \to U^T$.*

Proof. Let $U^T \in \hat{G}_\theta$ and $U^{\mathscr{S}} = \{U^S : S \in \mathscr{S}\} \subseteq \hat{G}_\theta$, where $T \in \mathcal{M}_L$ and $\mathscr{S} \subseteq \mathcal{M}_L$. Suppose U^T is weakly contained in $U^{\mathscr{S}}$. We will show that T is weakly contained in $\mathscr{S}$ (see Lemma 2.1). By [7, p. 371], $U^T|H_L$ is weakly contained in $U^{\mathscr{S}}|H_L = \{U^S|H_L : S \in \mathscr{S}\}$. But $U^T|H_L$ is weakly equivalent to $V^{T,G}$ and $U^{\mathscr{S}}|H_L$ is weakly equivalent to $V^{\mathscr{S},G} = \bigcup \{V^{S,G} : S \in \mathscr{S}\}$ (Lemma 3.4). Hence, $V^{T,G}$ is weakly contained in $V^{\mathscr{S},G}$ and, in particular, $T = V^{T,e}$ is weakly contained in $V^{\mathscr{S},G}$. Consider the set $\{V^{S,x} : S \in \mathscr{S}, x \in H_L\}$. This is precisely $\mathscr{S}$ itself since each element of H_L leaves each $S \in \mathscr{S}$ unchanged. Let $\mathscr{R} = V^{\mathscr{S},G} \sim \mathscr{S}$ (set difference) so that $V^{\mathscr{S},G} = \mathscr{R} \cup \mathscr{S}$ (disjoint). Thus, T is weakly contained in $\mathscr{R} \cup \mathscr{S}$ and belongs to the closure of $\mathscr{R} \cup \mathscr{S}$ in Rep (H_L) [12, Proposition 1.1]. Equivalently, T belongs to the closure of $\mathscr{R}$ or $\mathscr{S}$, i.e. T is weakly contained in $\mathscr{R}$ or $\mathscr{S}$. Suppose T is weakly contained in $\mathscr{R}$. Then $T|N$ is weakly contained in $\mathscr{R}|N = \{W|N : W \in \mathscr{R}\}$. But $T|N$ is a multiple of L and is therefore weakly equivalent to L. Each element of $\mathscr{R}$ is of the form $V^{S,y}$, where $S \in \mathscr{S}$, $y \in G$ and $y \notin H_L$. Also, $V^{S,y}|N$ is weakly equivalent to $\{(S^y|N)^x : x \in H_L\}$ [10, Proposition 2]. We may easily verify that $S^y|N$ is a multiple of L^y, so that $S^y|N$ is weakly equivalent to L^y. Thus, $(S^y|N)^x$ is weakly equivalent L^{yx}, $x \in H_L$ and $V^{S,y}|N$ is weakly equivalent to $\{L^{yx} : x \in H_L\}$. Therefore, $\mathscr{R}|N$ is weakly equivalent to $\theta' = \{L^{yx} : y \notin H_L, x \in H_L\}$. Finally, we obtain that L is weakly contained in θ' which is precisely $\theta \sim \{L\}$, i.e. $\theta = \theta' \cup \{L\}$ (disjoint), so that L belongs to the relative closure of θ' in θ. However, θ is homeomorphic to G/H_L [15, Theorem 1] which is discrete. Thus, θ' is relatively closed in θ and we have a contradiction. Consequently, T must be weakly contained in $\mathscr{S}$.

If V is any representation, let supp (V) denote its support [5, Definition 3.4.6].

If G/N is an R-group (i.e. every element of $\hat{G}$ is weakly contained in the regular representation of G/N), then L. Baggett has shown [2, Theorem 4.1A] that $\hat{G}_\theta$ is weakly contained in U^L, i.e. $\hat{G}_\theta \subseteq$ supp (U^L). The next theorem determines supp (U^L) precisely under a slightly weaker hypothesis.

THEOREM 3.2. *Let the notation be as in Mackey's Theorem. If H_L/N is an R-group, then* supp $(U^L) = p^{-1}(\text{cl}(\theta))$.

Proof. Let $V \in \hat{G}$, $p(V) = \lambda \in \hat{N}/G$ and $J \in \lambda$. Then there exists $T \in \mathcal{M}_J$ such that $U^T = V$. By [9, Theorem 4.3], V is weakly contained in U^L if and only if $T|N$ is weakly contained in $\{L^y : y \in G\}$. But $T|N$ is weakly equivalent to J. Thus, V is weakly contained in U^L if and only if J is weakly contained in $\{L^y : y \in G\}$ or equivalently, $\lambda \in \text{cl}(\theta)$ in $\hat{N}/G$. Hence, V is weakly contained in U^L if and only if $p^{-1}(\lambda) \subseteq p^{-1}(\text{cl}(\theta))$, i.e. $V \in p^{-1}(\text{cl}(\theta))$.

COROLLARY 1. *$\hat{G}_\theta$ is weakly contained in U^L.*

COROLLARY 2. *The support of U^L is $\hat{G}_\theta$ if and only if θ is a closed point in $\hat{N}/G$.*

The next theorem characterizes the closed unions of fibres in a useful particular case.

THEOREM 3.3. *Let the notation be as in Mackey's Theorem. Suppose G/N is an R-group and points are closed in $\hat{N}/G$. If $E \subseteq \hat{N}/G$, then $p^{-1}(\text{cl}(E)) = \text{cl}(p^{-1}(E))$.*

Proof. Since p is continuous, we need only show the forward inclusion. Let $V \in p^{-1}(\text{cl}(E))$. Then $\theta = p(V)$ is an element of $\text{cl}(E)$ and there exists $L \in \theta$, $T \in \mathcal{M}_L$ such that $U^T = V$. The representation $T|N$ is weakly equivalent to L. If $r: \hat{N} \to \hat{N}/G$ denotes the canonical mapping, then L is weakly contained in $r^{-1}(E)$ since $\theta \in \text{cl}(E)$. Hence, $T|N$ is weakly contained in $r^{-1}(E)$ and V is weakly contained in $\{U^J : J \in r^{-1}(E)\}$ [9, Theorem 4.3]. However, each U^J is weakly equivalent to $G_{\theta(J)}$, where $\theta(J)$ is the orbit of J (Theorem 3.2, Corollary 2), so that

$$\{U^J : J \in r^{-1}(E)\}$$

is weakly equivalent to $p^{-1}(E)$. Thus, V is weakly contained in $p^{-1}(E)$, i.e. $V \in \text{cl}(p^{-1}(E))$ and the proof is complete.

This theorem says precisely that (under the given hypotheses) a union of fibres in $\hat{G}$ is closed if and only if the underlying union of orbits in $\hat{N}$ is closed. Note also that if points are closed in $\hat{N}/G$, then necessarily this is the case for $\hat{N}$, i.e. N is CCR.

4. **Separation of dual points.** It will be useful for us to have some machinery for showing that certain groups are CCR while their duals are not Hausdorff.

PROPOSITION 4.1. *Suppose N is a closed normal (proper) subgroup of the group G such that G/N is an R-group. Let I be the identity representation of N, R the regular representation of G/N, $\pi: G \to G/N$ the canonical homomorphism and*

$$\hat{G}_1 = \{M\pi : M \in (G/N)^\wedge\}.$$

If U^I (equivalently $R\pi$) belongs to the inner hull-kernel closure of $\hat{G}' = \hat{G} \sim \hat{G}_1$ in Rep (G), *then $\hat{G}$ is not Hausdorff. (This result is true for nonseparable G as well in which case* Rep (G) *must be suitably defined. See* [9, p. 242].)

Proof. By hypothesis, there exists a sequence $\{T^i\}$ in $\hat{G}'$ such that $T^i \to U^I$ in Rep (G). Note that $U^I = R\pi$. Since G/N is an R-group, it follows that $\hat{G}_1$ is weakly contained in $R\pi$. Hence, $\{T^i\}$ converges to every element of $\hat{G}_1$ [12, Proposition 1.3]. But there exist at least two distinct elements in $\hat{G}_1$ [5, Corollaire 13.6.6], so that $\hat{G}$ is not Hausdorff.

REMARK. It is possible for U^I to be weakly contained in $\hat{G}'$ but not an element of its closure in Rep (G) (compare this with [12, Proposition 1.1]). Let N (non-compact) and H be two groups with Hausdorff duals and H an R-group. Then $G=N\times H$ has a Hausdorff dual; in fact $\hat{G}=\hat{N}\times\hat{H}$. In this case, $\hat{G}_1=\{I\}\times\hat{H}$ and $\hat{G}'=\hat{N}'\times\hat{H}$, where $\hat{N}'=\hat{N}\sim\{I\}$. By Proposition 4.1, U^I is not an accumulation point of $\hat{G}'$ in Rep (G). However, U^I is weakly contained in $\hat{G}'$ as the following argument verifies. Since N is not compact, I is not square integrable [5, p. 276] and hence, $\{I\}$ is not open in $\hat{N}$ [6, Proposition 3]. Therefore, $\hat{N}'$ is dense in $\hat{N}$, i.e. I is weakly contained in $\hat{N}'$ and consequently, U^I is weakly contained in $\{U^L : L\in\hat{N}'\}$ [9, Theorem 4.2]. But each U^L is weakly equivalent to $\{L\}\times\hat{H}$ (Theorem 3.2, Corollary 2), $L\in\hat{N}'$. Thus, U^I is weakly contained in $\hat{N}'\times\hat{H}=\hat{G}'$.

We now turn to the subject of CCR representations.

THEOREM 4.1. *Let G be a group, H a closed subgroup and T a representation of H. If T is CCR and G/H is compact, then U^T is CCR.*

Outline of Proof. The following is patterned after [1, p. 185] (see [22, Theorem 23] for a complete proof). Let Δ_G (respectively Δ_H) be the modular function for G (respectively H). Then there exists [4, p. 56] a continuous function $\rho: G\to(0,\infty)$ such that $\rho(hx)=[\Delta_H(h)/\Delta_G(h)]\rho(x)$, $x\in G$, $h\in H$. Let μ be the corresponding quasi-invariant measure on G/H (which is finite). Let dx and dh denote right Haar measures for G and H respectively. Suppose φ is a (nonzero) continuous complex function on G with compact support. It suffices to show U^T_φ is a compact operator. If $f\in\mathscr{H}(U^T)$, then [18, p. 107]

$$U^T_\varphi f(y) = \int_G \varphi(x)[\rho(yx)/\rho(y)]^{1/2}f(yx)\,dx, \qquad y\in G.$$

By a lengthy but straightforward computation, we may verify that

$$U^T_\varphi f(y) = \int_{G/H} k_\varphi(y,x)f(x)\,d\mu(Hx), \tag{1}$$

where

$$k_\varphi(y,x) = \int_H \varphi^*(y,x,h)T_h\,dh,$$

$$\varphi^*(y,x,h) = \Delta_G(y)\varphi(y^{-1}hx)[\Delta_G(h)/\rho(y)\rho(x)\Delta_H(h)]^{1/2},$$

and the integrand in (1) depends only on the right coset Hx to which x belongs. For each $x, y\in G$, $k_\varphi(y,x)$ is a compact operator on $\mathscr{H}(T)$ because T is CCR and the function $\varphi^*(y,x,\cdot)$ is continuous on H with compact support. Let $\mathscr{C}$ denote the Banach space of compact linear operators on $\mathscr{H}(T)$ with the usual norm topology.

PROPOSITION 4.2. *The mapping $k_\varphi: G\times G\to\mathscr{C}$ is continuous.*

Proof. This is a direct consequence of the continuity of the functions Δ_G, Δ_H and ρ together with the fact that φ is uniformly continuous [20, p. 63].

It is important to observe that the proof of this proposition does not require that G/H be compact or that T be CCR.

Now let $s: G/H \to G$ be a Borel cross-section such that $B=s(G/H)$ is relatively compact in G [18, Lemma 1.1]. (Note that there exists a compact subset F of G such that $\{Hx : x \in F\}=G/H$ [23, p. 19].) Hence, $k_\varphi(B\times B)$ is a norm-bounded subset of $\mathscr{C}$. Therefore, the composite function $k'_\varphi: (a, b) \to k_\varphi(s(a), s(b))$ is a bounded measurable function of $G/H\times G/H$ into $\mathscr{C}$, i.e.

$$k'_\varphi \in L^2(G/H\times G/H, \mu\times\mu, \mathscr{C}).$$

Now let $\mathscr{H}(U^T)$ be identified with $L^2(G/H, \mu, \mathscr{H}(T))$ by restricting each f in $\mathscr{H}(U^T)$ to B. We then obtain a linear operator K_φ on $L^2(G/H, \mu, \mathscr{H}(T))$ given by

$$K_\varphi f(a) = \int_{G/H} k'_\varphi(a, b)f(b)\, d\mu(b), \qquad a \in G/H,$$

which is compact [22]. Hence, U^T_φ is compact.

This theorem is essentially known for T equal to the identity representation of H [13]. Also, the compactness of G/H is not necessary for U^T to be CCR (see Proposition 5.4 of [1, p. 156]).

PROPOSITION 4.3. *Let G be a group and N a closed normal subgroup. If N is CCR and G/N is compact, then G is CCR.*

Proof. N is clearly type I [14]. From the hypotheses, it follows that the points of the orbit space $\hat{N}/G=\hat{N}/(G/N)$ are closed [21, Proposition 1.1.1], so that N is regularly embedded in G [15, Theorem 1]. This proposition is then a consequence of Mackey's Theorem, Theorem 4 of [1, pp. 176–177] and Theorem 4.1.

5. **The essentially free case.**

In [9, Theorem 5.2], Fell completely described the topology of $\hat{G}$ for G the Euclidean group of the plane. The most important single property of this group is that all of its nonidentity stability subgroups are the same. This phenomenon, together with more or less straightforward modifications in Fell's methods, will enable us to describe the dual topologies of a significant class of groups satisfying the hypotheses of Mackey's Theorem.

DEFINITION 5.1. Let G and N be as in Mackey's Theorem. Then (G, N) is *essentially free* if the stability subgroups H_L, $L \in \hat{N}$, $L\neq I$ are all the same, i.e. $H_L=H$ for all such L. (The stability subgroup of the identity is always G.) If, in particular, $H=N$, then we say (G, N) is *free*. Note that H is a closed normal subgroup of G containing N.

Throughout this section we will assume (G, N) is essentially free with H as above.

LEMMA 5.1. *N is regularly embedded in H so that Mackey's Theorem can be applied to (H, N). In fact, the stability subgroup in H of any $L \in \hat{N}$ is H itself. Hence, orbits are singletons and $\hat{N}/H=\hat{N}$.*

Consequently, we see that the fibre in $\hat{H}$ over the element L (i.e. the orbit $\{L\}$) in $\hat{N}$ is precisely $\mathscr{M}_L=\{L' \otimes M\pi_L : M \in (H/N, \sigma_L)\hat{}\}$. In particular, as always, the identity fibre $\hat{H}_1$ is $\{M\pi_I : M \in (H/N)\hat{}\}$. Let $\hat{H}'=\hat{H}\sim\hat{H}_1$. Since H is normal in G, we have the transformation group $G\times\hat{H}\to\hat{H}$ which decomposes into the two subtransformation groups $G\times\hat{H}_1\to\hat{H}_1$ and $G\times\hat{H}'\to\hat{H}'$ ($\hat{H}_1$ and $\hat{H}'$ are unions of orbits in $\hat{H}$). Thus, we may speak of the orbit spaces $\hat{H}_1/G$ and $\hat{H}'/G$. Furthermore, $\hat{H}_1/G$ (respectively $\hat{H}'/G$), viewed as a subset of $\hat{H}/G$, is closed (respectively open) since $\hat{H}_1$ is a closed subset of $\hat{H}$ [9, Lemma 5.2]. Let $\hat{G}_1$ denote the fibre in $\hat{G}$ over the identity element of $\hat{N}/G$ and $\hat{G}'=\hat{G}\sim\hat{G}_1$. As above, $\hat{G}_1$ is closed and $\hat{G}'$ is open in $\hat{G}$. If $S, T\in\hat{H}'$, then $U^T=U^S$ if and only if T and S belong to the same orbit in $\hat{H}'/G$ (i.e. in $\hat{H}/G$). (This is a consequence of Mackey's Theorem in the present case.) Thus, U^T depends only on the orbit θ in $\hat{H}'/G$ to which T belongs; denote it by U^θ. Then $\theta\to U^\theta$ is a bijection between $\hat{H}'/G$ and $\hat{G}'$. If $E\subseteq\hat{H}'/G$, we denote the closure of E in $\hat{H}/G$ by $\mathrm{cl}\,(E)$ and the relative closure of E in $\hat{H}'/G$ by $\mathrm{cl}'\,(E)$. Of course, $\mathrm{cl}'\,(E)=\mathrm{cl}\,(E)\cap\hat{H}'/G$. Also, we define $U^E=\{U^\theta : \theta\in E\}$ and let $t\colon\hat{H}\to\hat{H}/G$ be the canonical mapping.

LEMMA 5.2. *Let $E\subseteq\hat{H}'/G$ and $\theta\in\hat{H}'/G$. The following are equivalent.*

(i) *U^θ is weakly contained in U^E.*
(ii) *T is weakly contained in $t^{-1}(E)$, $T\in\theta$.*
(iii) *$t^{-1}(\theta)$ is weakly contained in $t^{-1}(E)$.*
(iv) *$\theta\in\mathrm{cl}\,(E)$.*
(v) *$\theta\in\mathrm{cl}'\,(E)$.*

Proof. (i) is equivalent to (ii) by [9, Theorem 4.4]. (ii) is equivalent to (iii) because the closure of a union of orbits is again a union of orbits. The remaining two equivalences are obvious.

PROPOSITION 5.1. *$\hat{H}'/G$ is homeomorphic to $\hat{G}'$ under the mapping $\theta\to U^\theta$.*

Proof. This is a consequence of the equivalence of (i) and (v) in the previous lemma.

We will find it necessary to assume also that G/H is an R-group for the remainder of this section. Accordingly, it will be convenient to say that (G, N) is *admissible* if it is essentially free and G/H is an R-group.

Let $F\subseteq\hat{H}_1/G$ and define

$$\hat{G}_1(F) = \{V\in\hat{G}_1 : t(\mathrm{supp}\,(V|H)) = F\}.$$

LEMMA 5.3. *Suppose (G, N) is admissible. Let $E\subseteq\hat{H}'/G$ and $F\subseteq\hat{H}_1/G$.*

(i) *If $F\subseteq\mathrm{cl}\,(E)$, then every element of $\hat{G}_1(F)$ is weakly contained in U^E.*
(ii) *If $F\nsubseteq\mathrm{cl}\,(E)$, then no element of $\hat{G}_1(F)$ is weakly contained in U^E.*

Proof. Suppose $V \in \hat{G}_1(F)$. By [9, Theorem 4.3], V is weakly contained in U^E if and only if $V|H$ is weakly contained in $t^{-1}(E)$, i.e. $t(\text{supp}(V|H)) \subseteq \text{cl}(E)$.

Let q be the inverse of the mapping in Proposition 5.1.

THEOREM 5.1. *Let* (G, N) *be admissible and* $\mathcal{S} \subseteq \hat{G}$. *Then* $\mathcal{S}$ *is closed if and only if*

(i) $\mathcal{S} \cap \hat{G}_1$ *is closed in* $\hat{G}_1$,

(ii) $q(\mathcal{S} \cap \hat{G}')$ *is relatively closed in* $\hat{H}'/G$ *and*

(iii) *if* $F \subseteq \hat{H}_1/G$ *and* $F \subseteq \text{cl}(q(\mathcal{S} \cap \hat{G}'))$, *then* $\hat{G}_1(F) \subseteq \mathcal{S}$.

Proof. If $\mathcal{S}$ is closed, then (i) and (ii) are obvious. Part (iii) follows from Lemma 5.3(i) with $E = q(\mathcal{S} \cap \hat{G}')$. (Note that $U^E = \mathcal{S} \cap \hat{G}'$.)

Suppose (i), (ii) and (iii) hold. Then $\mathcal{S} \cap \hat{G}_1$ is closed in $\hat{G}$, because $\hat{G}_1$ is closed in $\hat{G}$. Since $\mathcal{S} = (\mathcal{S} \cap \hat{G}_1) \cup (\mathcal{S} \cap \hat{G}')$, it remains to show that if $V \in \hat{G}$ and V is weakly contained in $\mathcal{S} \cap \hat{G}'$, then $V \in \mathcal{S}$. For convenience, let $E = q(\mathcal{S} \cap \hat{G}')$. If $V \in \hat{G}'$, then $V = U^\theta$, for some $\theta \in \hat{H}'/G$. By hypothesis, V belongs to the relative closure of $\mathcal{S} \cap \hat{G}'$ in $\hat{G}'$. Hence, $\theta \in \text{cl}'(E)$ (Proposition 5.1). By (ii), $\theta \in E$ and $U^\theta \in U^E = \mathcal{S} \cap \hat{G}' \subseteq \mathcal{S}$. If $V \in \hat{G}_1$, let $F = t(\text{supp}(V|H))$. Then $F \subseteq \hat{H}_1/G$ and $V|H$ is weakly contained in $\{U^\theta|H : \theta \in E\}$ [7, p. 371]. But each $U^\theta|H$ is weakly equivalent to $t^{-1}(\theta)$, $\theta \in E$, so that $V|H$ is weakly contained in $t^{-1}(E)$. Hence, $\text{supp}(V|H) \subseteq \text{cl}(t^{-1}(E))$ and $F \subseteq \text{cl}(E)$. By (iii), $\hat{G}_1(F) \subseteq \mathcal{S}$ and, in particular, $V \in \mathcal{S}$.

COROLLARY. *If* (G, N) *is admissible and free* (*i.e.* $H = N$), *then* $\mathcal{S}$ *is closed if and only if*

(i) $\mathcal{S} \cap \hat{G}_1$ *is closed in* $\hat{G}_1$,

(ii) $p(\mathcal{S} \cap \hat{G}')$ *is relatively closed in* $\hat{N}'/G$, *and*

(iii) *if the identity orbit of* $\hat{N}/G$ *belongs to* $\text{cl}(p(\mathcal{S} \cap \hat{G}'))$, *then* $\hat{G}_1 \subseteq \mathcal{S}$.

Proof. In this case, $p = q$ on $\hat{G}'$ and $\hat{H}_1 = \hat{N}_1$ consists only of the identity orbit.

Observe that Proposition 2.1 of [1, p. 172] is an application of this corollary.

PROPOSITION 5.2. *Let* (G, N) *be admissible. Suppose* (i) $H \neq G$, (ii) G *is an* R-*group*, (iii) N *is not compact and* (iv) N *is CCR. Then* $\hat{G}$ *is not Hausdorff.*

Proof. From (ii) it follows that H, G/N, N and H/N are R-groups. By (iii), the identity I of $\hat{N}$ is not square integrable and thus, $\hat{N}' = \hat{N} \sim \{I\}$ is dense in $\hat{N}$ (see the remark following Proposition 4.1). Consequently, $\hat{H}'$ is dense in $\hat{H}$ (Theorem 3.3 applied to (H, N) with $E = \hat{N}'$). Therefore, there exists a sequence $\{T^i\}$ in $\hat{H}'$ converging to the identity J of $\hat{H}$. By [12, Proposition 1.2] and [9, Theorem 4.2], the sequence $\{U^{T^i}\}$ converges to U^J in $\text{Rep}(G)$. But each U^{T^i} is irreducible, i.e. U^J is an accumulation point of $\hat{G}'$ in $\text{Rep}(G)$. Now apply Proposition 4.1.

6. **Compact extensions of abelian groups.** Throughout this section we will consider the class of separable locally compact groups G having a closed normal abelian subgroup A (depending on G) for which G/A is compact. Denote this collection by $[CA]$. Of course, $[CA]$ contains the (separable) compact and abelian groups. Recall that abelian groups are type I. Note that A is regularly embedded in G.

Proposition 6.1. *Let $\theta \in \hat{A}/G$ and $\chi \in \theta$. Then the support of U^{χ} is $\hat{G}_{\theta}$.*

Proof. See Corollary 2 of Theorem 3.2. Note that $\hat{A}/G$ is a Hausdorff space (this is proved as in [21, Proposition 1.1.8]).

Lemma 6.1. *For each $\chi \in \hat{A}$, $\mathscr{M}_{\chi}$ is countable and discrete in its relative topology.*

Proof. See Proposition 3.2.

Proposition 6.2. *Let $\theta \in \hat{A}/G$, $\chi \in \theta$. If H_{χ} is either open or normal, then $\hat{G}_{\theta}$ is countable and relatively discrete.*

Proof. Combine Theorem 3.1 and [10, Lemma 4] with Lemma 6.1.

Proposition 6.3. *Let $E \subseteq \hat{A}/G$ and denote $p^{-1}(E)$ by $\hat{G}_E$. Then $\hat{G}_E$ is closed in $\hat{G}$ if and only if E is closed in $\hat{A}/G$.*

Proof. This is Theorem 3.3 for $[CA]$-groups.

Proposition 6.4. *$[CA]$ groups are CCR (i.e. points are closed in $\hat{G}$) and hence, type* I.

Proof. Apply Proposition 4.3.

This proposition includes the corresponding theorem of Baggett [3, p. 198] when the $[CA]$-group G is a semidirect product of A with G/A.

Lemma 6.2. *If G is a central group, then $\hat{G}$ is Hausdorff.*

Proof. This result is due to Grosser and Moskowitz [17].

Lemma 6.3. *If $G \in [CA]$ and V, W are two elements of $\hat{G}$ belonging to different fibres, i.e. $p(V) \neq p(W)$, then V and W can be separated by disjoint open subsets of $\hat{G}$.*

Proof. Recall that p is continuous.

This, however, is as far as the Hausdorff property extends. There are many examples of noncentral groups in $[CA]$ which have non-Hausdorff duals. We do not know if it is possible for there to exist a noncentral group in $[CA]$ with Hausdorff dual. See Proposition 6.6 below and Proposition 9.1-B of [3, p. 211].

We now apply §5 to $[CA]$-groups. Suppose the $[CA]$-group (G, A) is essentially free. Since G/A is compact, (G, A) is automatically admissible. The remaining notation is as in §5.

Proposition 6.5. *Let (G, A) be essentially free. If $\mathscr{S} \subseteq \hat{G}$, then $\mathscr{S}$ is closed if and only if*

(i) *$q(\mathscr{S} \cap \hat{G}')$ is (relatively) closed in $\hat{H}'/G$ and*

(ii) *if $\lambda \in \hat{H}_1/G$ and $\lambda \in \mathrm{cl}\,(q(\mathscr{S} \cap \hat{G}'))$, then $\hat{G}_1(\lambda) \subseteq \mathscr{S}$.*

Proof. The fibre $\hat{G}_1$ is relatively discrete (Proposition 6.2) and closed in $\hat{G}$. Hence, (i) of Theorem 5.1 is automatically satisfied. Furthermore, it is clear that

$(G/N, H/N)$ satisfies the hypotheses of Mackey's Theorem, so that for each element V of $\hat{G}_1$, $V|H$ is weakly equivalent to some $t^{-1}(\lambda)$, $\lambda \in \hat{H}_1/G$ [10, Lemma 2]. Thus, if $F \subseteq \hat{H}_1/G$, then $\hat{G}_1(F) \neq \varnothing$ if and only if F is a singleton.

This proposition describes the topology of $\hat{G}$ for (G, A) essentially free in terms of the topology of $\hat{H}$, H a central group. The topological structure of such duals $\hat{H}$ has been determined by Grosser and Moskowitz [17]. Of course, Proposition 6.5 coincides with [3, Theorem 6.2-A] when the essentially free group (G, A) is a semidirect product.

Corollary 1. *If (G, A) is free, i.e. $H=A$, then $\mathcal{S} \subseteq \hat{G}$ is closed if and only if*
(i) *$p(\mathcal{S} \cap G')$ is (relatively) closed in $\hat{A}'/G$ and*
(ii) *if the identity orbit of $\hat{A}/G$ belongs to* cl $(p(\mathcal{S} \cap \hat{G}'))$, *then $\hat{G}_1 \subseteq \mathcal{S}$.*

Proposition 6.6. *$\hat{G}$ is Hausdorff if and only if $H=G$, i.e. G is a central group.*

Proof. Apply Proposition 5.2 and Lemma 6.2. (Note that $[CA]$ groups are R-groups).

We conclude with an example of the essentially free case. Let T denote the circle group and Z the group of integers. Let A be the field of complex numbers and $Q=T \times T$. Let G be the semidirect product $A \cdot Q$ with product topology and group operation given by

$$(a, t)(b, s) = (a+tb, ts), \qquad a, b \in A, \quad s, t \in Q,$$

where $t=(t_1, t_2)$ and $tb=t_1t_2b$ (ordinary complex multiplication). Then $\hat{A}=A$ and each $\chi \in \hat{A}$ is of the form χ_α, for α a complex number, where

$$\chi_\alpha(a) = \exp(i \operatorname{Re}(\bar{\alpha}a)), \qquad a \in A.$$

Let α be a nonzero element of A. If $y \in G$ and $y=(a, t)$ then

$$\chi_\alpha^y(a) = \chi_\alpha(t_1t_2a) = \exp(i \operatorname{Re}(t_1t_2a\bar{\alpha})), \qquad a \in A.$$

It is not difficult to verify that

$$H_\alpha = \{(a, t) : t = (t_1, \bar{t}_1), a \in A, t_1 \in T\}.$$

Observe that H_α does not depend on α as long as $\alpha \neq 0$. Hence, for each $\alpha \in A$, $\alpha \neq 0$, we have $H_\alpha = H$ as above. Therefore, (G, A) is essentially free (in fact, admissible) but not free.

Lemma 6.4. *The group H is topologically isomorphic to $A \times T$, i.e. it is abelian.*

Hence, $\hat{H}=\hat{A}\times\hat{T}=A\times Z$ and each $\gamma \in \hat{H}$ is of the form $\gamma=(\gamma_\alpha, \gamma_n)$, $\alpha \in A$, $n \in Z$, where

$$\gamma(a, (t, \bar{t})) = \gamma_\alpha(a)\gamma_n(t) = t^n \exp(i \operatorname{Re}(a\bar{\alpha})), \qquad a \in A, \quad t \in T.$$

It is worthwhile remarking that if we replace A by H, then the resulting $[CA]$-group (G, H) is no longer essentially free (the stability subgroup of an element of $\hat{H}_1$ is all of G). Our present approach enables us to apply Theorem 5.1.

Let $y=(b,(s_1,s_2))$ be an element of G. Then $\gamma^y=(\gamma_\delta,\gamma_n)$, where $\delta=\alpha\overline{s_1s_2}$, for $\gamma\in\hat{H}$ as above. The orbit of γ in $\hat{H}$ due to the action of G is

$$\theta(\gamma) = \{(\gamma_\beta,\gamma_n) : |\beta| = |\alpha|\},$$

so that this action is constant on $\hat{T}=Z$. Therefore,

$$\hat{H}/G = \hat{A}/G\times\hat{T} = [0,\infty)\times Z,$$

with $\hat{H}'/G=(0,\infty)\times Z$ and $\hat{H}_1/G=\{0\}\times Z$. We know that $\hat{G}_1=(G/A)^\wedge=Z\times Z$. We also know that $\hat{G}'$ is homeomorphic to $\hat{H}'/G$. This homeomorphism q is given by

$$U^{\gamma_\alpha\gamma_n}\to(\gamma_\alpha,\gamma_n), \qquad (\alpha,n)\in(0,\infty)\times Z.$$

If $\lambda=(0,n)\in\hat{H}_1/G$, then $\hat{G}_1(\lambda)=\hat{G}_1(0,n)=Z\times\{n\}$. Let $\mathscr{S}$ be a subset of $\hat{G}$. Then $\mathscr{S}\cap\hat{G}_1\subseteq Z\times Z$ is automatically closed and $q(\mathscr{S}\cap G')\subseteq(0,\infty)\times Z$.

PROPOSITION 6.7. *$\mathscr{S}$ is closed if and only if*
(i) *$q(\mathscr{S}\cap\hat{G}')$ is closed in $(0,\infty)\times Z$ and*
(ii) *if the orbit $(0,n)$ belongs to the closure of $q(\mathscr{S}\cap\hat{G}')$ in $[0,\infty)\times Z$, then the fibre $Z\times\{n\}$ (in $\hat{G}_1$) is contained in $\mathscr{S}$.*

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